Orthogonal Factorization Systems for Double Categories

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Orthogonal factorization systems in a 1-category provide a notion of images along arrows. Various factorization systems on the category of categories give rise to important notions of images for functors, such as replete images, essential images and full images. In general, orthogonal factorization systems reveal important properties of the structure of a category, and may assist in defining a functor on a category.

In this talk, we will generalize orthogonal factorization systems to the double-categorical setting. We call these *double orthogonal factorization systems* (DOFS); this will allow us to define images of double cells in the direction of tight arrows. Although a notion of factorization for double cells was introduced in [5], no orthogonality conditions are imposed. In order to obtain a notion of DOFS, we will consider normal pseudo-category objects in a suitable 2-category of orthogonal factorization systems. However, normal pseudo-category objects can be viewed as models of a limit sketch with two types of arrows: tight arrows for domain, codomain, and identity, and a loose arrow for composition. We consider category objects in various enhanced 2-categories of orthogonal factorization systems. Classically, a good way to capture and understand orthogonal factorization systems and morphisms between them is in terms of algebras for a particular 2-monad. A result of Coppey [3] shows that strict factorization systems can be viewed as strict algebras for the squaring 2-monad (sending any category to its category of squares). The full result characterizing orthogonal factorization systems with a chosen factorization as the pseudo-algebras for this 2-monad was established by Korostenski and Tholen in [4]. This leads us to four natural notions of morphism between categories with an OFS: the strict algebra maps preserve the chosen factorizations, the pseudo maps preserve both classes of morphisms, the lax maps preserve the right class, and oplax maps preserve the left class. This provides us with three enhanced 2-categories in which to define a corresponding notion of DOFS. We illustrate this with several examples in each category of algebras. Several common types of double categories, including double categories of spans, relations, and modules, are shown to admit double factorization system under our definition. Further examples of DOFSs will be presented by R. Maldonado in a subsequent talk.

Another key aspect of orthogonal factorization systems is their interaction with Grothendieck fibrations: any fibration gives rise to a *Cartesian factorization system* on its domain, whose left class consists of the vertical arrows in the fibration (and satisfies the 3-for-2 property) and right class consists of Cartesian arrows. This result can be generalized to show that any orthogonal factorization system on the base of a fibration can be lifted to the total category canonically. We show that the recently-developed theory of double fibrations in [2] interacts analogously with our notion of orthogonal factorizations on double categories, allowing a DOFS to be lifted along a double fibration.

[1] C. Aberlé, E. Caviglia, M. Kukla, R. Maldonado, L. Mesiti, D. Pronk, and T. Ralaivaosoana *Double Orthogonal Factorization Systems*, preprint, arXiv:2509.26343, 2025.

[2] G. Crutwell, M. Lambert, D. Pronk, and M. Szyld, *Double Fibrations*, Theory and Applications of Categories, Vol. 38, No. 35, 2022, pp. 1326–1394.

[3] L. Coppey, *Algébres de decompositions et precategories*, PhD thesis, Université de Picardie Jules Verne, 1978.

[4] M. Korostenski and W. Tholen, *Factorization systems as Eilenberg-Moore algebras*, Journal of Pure and Applied Algebra **85**(1), pp. 57-72, 1993.

[5] C. Eberhard, T. Hirschowitz, and A. Laouar, *Template games, simple games, and Day convolution*, 4th International Conference on Formal Structures for Computation and Deduction, 2019.