Orthogonal Factorization Systems for Double Categories

Matthew Kukla [1]

The George Washington University

CT 2026

[1] Joint with C. Aberlé, E. Caviglia, R. Maldonado, L. Mesiti, D. Pronk, and T. Ralaivaosoana

Overview

Introduction

Orthogonal factorization systems provide a category with a notion of images along its arrows, generalizing the usual image factorization on categories such as **Set**.

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads, "\text{surjective}"'] & & Y \\
& \mathrm{Im}(X) \arrow[ur, hook, "\text{injective}"'] &
\end{tikzcd}
$$

Our goal is to extend such a notion to the double-categorical setting. We call these **double orthogonal factorization systems (DOFS)**. This allows us to define images of double cells in the direction of tight arrows.

Orthogonal factorization systems

Definition

An **orthogonal factorization system** (**OFS**) on a category $\mathcal{C}$ consists of two classes of arrows $\mathcal{L}, \mathcal{R}$ such that:

- Any arrow $f : A \to B$ can be written as $f = r_f \circ \ell_f$, where $r_f \in \mathcal{R}, \ell_f \in \mathcal{L}$
- $\mathcal{L}, \mathcal{R}$ are closed under composition, and contain all isomorphisms
- For any commutative square with $u \in \mathcal{L}, v \in \mathcal{R}$, there exists a unique e making the following commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, two heads, "u"'] & A' \arrow[d, hook, "v"] \\ B \arrow[r, "g"'] \arrow[ur, dotted, "\exists!\ e"] & B' \end{tikzcd}$$

Orthogonal factorization systems

Notable examples include:

- (surjections, injections) on **Set**
- (regular epimorphisms, monomorphisms) on any regular category
- for a fibration $p : \mathcal{E} \to \mathcal{B}$, the classes of (vertical, cartesian) morphisms on $\mathcal{E}$
- (bijective-on-objects, fully faithful) functors on **Cat**
- (final functors, discrete fibrations) on **Cat**

Orthogonal factorization systems

Lemma

In a category with an OFS, for f, g with factorizations below, there exists $e : \mathrm{Im}(f) \to \mathrm{Im}(g)$ as below:

$$\begin{tikzcd}
A \arrow[r, two heads, "\ell_f"'] \arrow[d, "u"'] \arrow[rr, bend left, "f"] & \mathrm{Im}(f) \arrow[r, hook, "r_f"'] \arrow[d, dotted, "\exists\ e"] & B \arrow[d, "v"] \\
A' \arrow[r, two heads, "\ell_g"] \arrow[rr, bend right, "g"'] & \mathrm{Im}(g) \arrow[r, hook, "r_g"] & B'
\end{tikzcd}$$

Double categories

Definition

A double category $\mathbb{A}$ consists of:

- A collection $\mathbb{O}\mathrm{b}(\mathbb{A})$ of objects
- Two distinct types of arrows between objects:
  - **Arrows**, $f : A \to B$, with (associative) composition $g \circ f$ and identities $1_A : A \to A$
  - **Proarrows**, $u : A \nrightarrow B$, with (possibly weak) composition $v \otimes u$ and identities $1_A^{\otimes} : A \nrightarrow A$
- **Double cells**, framed by pairs of arrows/proarrows, which may be composed vertically or horizontally

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"', "|" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v", "|" marking] \\
A' \arrow[r, "g"'] & B'
\end{tikzcd}$$

Double categories

Equivalently, a double category is a weak category object in **Cat**.

$$\mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \xrightarrow{\otimes} \mathcal{A}_1 \overset{\xrightarrow{\text{src}}}{\underset{\xrightarrow[\text{tgt}]{}}{\xleftarrow{\ \text{i}\ }}} \mathcal{A}_0$$

- Objects of $\mathcal{A}_0$ are the objects of $\mathbb{A}$
- Arrows of $\mathcal{A}_0$ are the arrows of $\mathbb{A}$
- Objects of $\mathcal{A}_1$ are the **proarrows** of $\mathbb{A}$
- Arrows of $\mathcal{A}_1$ are **double cells** of $\mathbb{A}$
- Composition of cells is determined by $\otimes$

Double orthogonal factorization systems

What would a notion of OFS look like for double categories? Natural place to start: a weak category object internal to some 2-category of categories with an OFS.

Definition

Let $\mathcal{C}, \mathcal{D}$ be categories equipped with an OFS. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a:

- **pseudo-morphism** of categories with an OFS if F preserves both classes
- **lax morphism** of categories with an OFS if it preserves the right class
- **colax morphism** of categories with an OFS if it preserves the left class
- **strict morphism** of categories with an OFS when it preserves both classes, and factorizations up to equality

Double orthogonal factorization systems

Definition

Define the following 2-categories:

- $\mathcal{F}\mathrm{act}_{\mathrm{ps}}$, whose 1-cells are pseudo-morphisms of categories equipped with an OFS
- $\mathcal{F}\mathrm{act}_{\mathrm{lax}}$, whose 1-cells are lax morphisms
- $\mathcal{F}\mathrm{act}_{\mathrm{colax}}$, whose 1-cells are colax morphisms
- $\mathcal{F}\mathrm{act}_{\mathrm{str}}$, whose 1-cells are strict morphisms

In all cases, 2-cells are ordinary natural transformations.

Double orthogonal factorization systems

Definition

A pseudo (resp. lax, colax, strict) **double orthogonal factorization system** (or DOFS) is a pseudo-category internal to $\mathcal{F}\mathrm{actps}$ (resp. $\mathcal{F}\mathrm{act}_{\mathrm{lax}}, \mathcal{F}\mathrm{act}_{\mathrm{colax}}, \mathcal{F}\mathrm{act}_{\mathrm{str}}$) such that $\mathrm{src}, \mathrm{tgt}$, and i preserve both classes of an OFS.

In other words, we have orthogonal factorization systems on $\mathcal{A}_0$ and $\mathcal{A}_1$ which are compatible.

Example: $\mathbb{S}\mathrm{pan}(\mathcal{C})$

Suppose $\mathcal{C}$ is a category with pullbacks. The double category $\mathbb{S}\mathrm{pan}(\mathcal{C})$ has arrows of $\mathcal{C}$ as arrows, and spans in $\mathcal{C}$ as proarrows. A double cell α is given by a diagram of the form:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\shortmid" marking] & A' \arrow[d, "\shortmid" marking] \\
B \arrow[r, "g"'] \arrow[ur, phantom, "\alpha"] & B'
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
 & A \arrow[rr, "f"] & & A' \\
X \arrow[ur] \arrow[dr] \arrow[rr, "\alpha"] & & Y \arrow[ur] \arrow[dr] & \\
 & B \arrow[rr, "g"'] & & B'
\end{tikzcd}
$$

Example: $\mathbb{S}\mathrm{pan}(\mathcal{C})$

Given an OFS $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$, we can horizontally factor a cell in $\mathbb{S}\mathrm{pan}(\mathcal{C})$:

$$
\begin{tikzcd}
 & A \arrow[rr, "f"] & & A' \\
X \arrow[ur] \arrow[dr] \arrow[rr, "\alpha"] & & Y \arrow[ur] \arrow[dr] & \\
 & B \arrow[rr, "g"'] & & B'
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
 & A \arrow[rr, two heads, "\ell_f"] & & \mathrm{Im}(f) \arrow[rr, hook, "r_f"] & & A' \\
X \arrow[ur] \arrow[dr] \arrow[rr, two heads, "\ell_\alpha"] & & \mathrm{Im}(\alpha) \arrow[ur] \arrow[dr] \arrow[rr, hook, "r_\alpha"] & & Y \arrow[ur] \arrow[dr] & \\
 & B \arrow[rr, two heads, "\ell_g"'] & & \mathrm{Im}(g) \arrow[rr, hook, "r_g"'] & & B'
\end{tikzcd}
$$

This gives two classes of double cells $(\mathbb{L}, \mathbb{R})$ whose arrow components come from $\mathcal{L}, \mathcal{R}$ respectively.

Example: $\mathbb{S}\mathrm{pan}(\mathcal{C})$

Vertical composition of cells only preserves the right class. In $\mathbb{S}\mathrm{pan}(\mathbf{Set})$, we have the following counterexample:

$$
\begin{tikzcd}
\{s_0\} \arrow[r, two heads] & \{*\} \\
\{(s_0, t_0)\} \arrow[u, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[r, two heads] & \{*\} \arrow[u] \arrow[d] \\
\{t_0, t_1\} \arrow[r, two heads] & \{*\} \\
\{(t_1, s_1)\} \arrow[u, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[r, two heads] & \{*\} \arrow[u] \arrow[d] \\
\{s_1\} \arrow[r, two heads] & \{*\}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\{s_0\} \arrow[r, two heads] & \{*\} \\
\varnothing \arrow[u] \arrow[d] \arrow[r, red] & \{(*, *)\} \arrow[u] \arrow[d] \\
\{s_1\} \arrow[r, two heads] & \{*\}
\end{tikzcd}
$$

Let $\mathcal{C}$ be a regular category. The double category $\mathbb{R}\mathrm{el}(\mathcal{C})$ has objects of $\mathcal{C}$ as objects, arrows of $\mathcal{C}$ as arrows, and relations in $\mathcal{C}$ as proarrows. Cells in $\mathbb{R}\mathrm{el}(\mathcal{C})$ take the form:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "R"', "\shortmid" marking] \arrow[dr, phantom, "\alpha"] & A' \arrow[d, "R'", "\shortmid" marking] \\
B \arrow[r, "g"'] & B'
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
R \arrow[r, hook] \arrow[d, "\alpha"'] & A \times B \arrow[d, "f \times g"] \\
R' \arrow[r, hook] & A' \times B'
\end{tikzcd}
$$

Example: $\mathbb{R}\mathrm{el}(\mathcal{C})$

There exists a DOFS $(\mathbb{L}, \mathbb{R})$ on $\mathbb{R}\mathrm{el}(\mathcal{C})$ where:

- the left class $\mathbb{L}$ consists of cells with regular epimorphisms as arrows

$$\begin{tikzcd} A \arrow[r, two heads, "e_0"] \arrow[d, "R"', "\shortmid" marking] \arrow[dr, phantom, "\theta\in\mathbb{L}"] & A' \arrow[d, "R'", "\shortmid" marking] \\ B \arrow[r, two heads, "e_1"'] & B' \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} R \arrow[r, hook] \arrow[d, two heads, "\theta"'] & A \times B \arrow[d, two heads, "e_0 \times e_1"] \\ R' \arrow[r, hook] & A' \times B' \end{tikzcd}$$

- the right class $\mathbb{R}$ consists of cells with monomorphisms as arrows

$$\begin{tikzcd} A \arrow[r, hook, "m_0"] \arrow[d, "R"', "\shortmid" marking] \arrow[dr, phantom, "\psi\in\mathbb{R}"] & A' \arrow[d, "R'", "\shortmid" marking] \\ B \arrow[r, hook, "m_1"'] & B' \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} R \arrow[r, hook] \arrow[d, hook, "\psi"'] & A \times B \arrow[d, hook, "m_0 \times m_1"] \\ R' \arrow[r, hook] & A' \times B' \end{tikzcd}$$

Example: $\mathbb{R}\mathrm{el}(\mathcal{C})$

Cells factorize as:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "R"', "|" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "S", "|" marking] \\
A' \arrow[r, "g"'] & B'
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, two heads, "\ell_f"] \arrow[d, "R"', "|" marking] & \mathrm{Im(f)} \arrow[r, hook, "r_f"] \arrow[d, "\mathrm{Im}(\alpha)", "|" marking] & B \arrow[d, "S", "|" marking] \\
A' \arrow[r, two heads, "\ell_g"'] & \mathrm{Im(g)} \arrow[r, hook, "r_g"'] & B'
\end{tikzcd}
$$

The relation $\mathrm{Im}(\alpha)$ is induced by functoriality of the OFS on $\mathcal{C}$:

$$
\begin{tikzcd}
R \arrow[r, hook] \arrow[d, two heads, "\ell_\alpha"'] & A \times B \arrow[d, two heads, "\ell_f \times \ell_g"] \\
\mathrm{Im}(\alpha) \arrow[r, hook, dashed] \arrow[d, hook, "r_\alpha"'] & \mathrm{Im}(f) \times \mathrm{Im}(g) \arrow[d, hook, "r_f \times r_g"] \\
R' \arrow[r, hook] & A' \times B'
\end{tikzcd}
$$

Fibrations and orthogonal factorization systems

Fibrations of 1-categories interact with orthogonal factorization systems.

Theorem

Let $P : \mathcal{E} \to \mathcal{B}$ be a cloven Grothendieck fibration and let $(\mathcal{L}, \mathcal{R})$ be an OFS on $\mathcal{B}$. Then $(\mathcal{L}, \mathcal{R})$ lifts to an OFS $(\mathcal{L}^P, \mathcal{R}^P)$ on $\mathcal{E}$, where $\mathcal{L}^P$ is the collection of morphisms over $\mathcal{L}$ and $\mathcal{R}^P$ is the collection of cartesian morphisms over $\mathcal{R}$.

Corollary

Any Grothendieck fibration $p : \mathcal{E} \to \mathcal{B}$ gives an OFS consisting of (vertical, cartesian) morphisms on $\mathcal{E}$, where a morphism is **vertical** when it is sent to an isomorphism by p.

Double fibrations

Recall that a double fibration $P : \mathbb{E} \to \mathbb{B}$ between double categories is a pseudo-category in the category of fibrations. This amounts to a double functor $P : \mathbb{E} \to \mathbb{B}$ between pseudo double categories consisting of fibrations componentwise, with the additional requirement that $\mathrm{i}_{\mathbb{E}}, \mathrm{tgt}_{\mathbb{E}}$, and $\otimes_{\mathbb{E}}$ preserve cleavages.

$$\begin{tikzcd}
\mathcal{E}_1 \times_{\mathcal{E}_0} \mathcal{E}_1 \arrow[r, "\otimes_{\mathbb{E}}"] \arrow[d, "P_1 \times_{P_0} P_1"'] & \mathcal{E}_1 \arrow[r, shift left=2, "\mathrm{src}_{\mathbb{E}}"] \arrow[r, shift right=2, "\mathrm{tgt}_{\mathbb{E}}"'] \arrow[d, "P_1"] & \mathcal{E}_0 \arrow[l, "\mathrm{i}_{\mathbb{E}}" description] \arrow[d, "P_0"] \\
\mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \arrow[r, "\otimes_{\mathbb{B}}"'] & \mathcal{B}_1 \arrow[r, shift left=2, "\mathrm{src}_{\mathbb{B}}"] \arrow[r, shift right=2, "\mathrm{tgt}_{\mathbb{B}}"'] & \mathcal{B}_0 \arrow[l, "\mathrm{i}_{\mathbb{B}}" description]
\end{tikzcd}$$

Lifting through double fibrations

Theorem

Let $P : \mathbb{E} \to \mathbb{B}$ be a double fibration where the unit functor is cleavage-preserving, and let $(\mathbb{L}, \mathbb{R})$ be a pseudo double orthogonal factorization system (resp., a lax or colax one) on $\mathbb{B}$. Then $(\mathbb{L}, \mathbb{R})$ lifts to a pseudo double orthogonal factorization system (resp., a lax or colax one) $(\mathbb{L}^P, \mathbb{R}^P)$ on $\mathbb{E}$.

Lifting through double fibrations

Corollary

Let $P : \mathbb{E} \to \mathbb{B}$ be a double fibration where the units functor is cleavage-preserving. Then $\mathbb{E}$ has the pseudo DOFS (pseudo-vertical, cartesian), (pseudo-vertical, cartesian), where an arrow is said to be **pseudo-vertical** if P sends it to an isomorphism.